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Bell–Kochen–Specker theorem for any finite dimension $n \geqslant 3$

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Abstract. The Bell–Kochen–Specker theorem against non-contextual hidden variables can be proved by constructing a finite set of 'totally non-colourable' directions, as Kochen and Specker did in a Hilbert space of dimension $n = 3$. We generalize Kochen and Specker's set to Hilbert spaces of any finite dimension $n \geqslant 3$, in a three-step process that shows the relationship between different kinds of proofs ('continuum', 'probabilistic', 'state-specific' and 'state-independent') of the Bell–Kochen–Specker theorem. At the same time, this construction of a totally non-colourable set of directions in any dimension explicitly solves the question raised by Zimba and Penrose about the existence of such a set for $n = 5$.

1. Introduction

Kochen and Specker (KS) [1–4] and, independently, Bell [5] proved what nowadays is called the Bell–Kochen–Specker (BKS) theorem, implying that quantum mechanics (QM) cannot be 'completed' with non-contextual hidden variables (NCHV), i.e. those which assign definite values to physical observables independently of which other compatible observables are jointly measured. The proof of the BKS theorem is based in the following mathematical result. Given a Hilbert space of dimension n, one can find sets of rays that cannot be mapped to the two-element set $\{0, 1\}$ in such a way that for *any* subset of n mutually orthonormal rays, the images of $n - 1$ of them are 0 and the image of the remaining ray is 1. Such sets (which in this paper shall be called *totally non-colourable sets,* TNCSs) exist in any Hilbert space of dimension $n \geqslant 3$; in particular, KS [4] produced an example with 117 rays in a real Hilbert space with $n = 3$. The connection between the impossibility of NCHV and the existence of TNCSs will be briefly reviewed in section 2 of this paper.

Recently, Zimba and Penrose [6] have proved that combining two TNCSs in dimensions n and m one can obtain a new TNCS in dimension $n + m$. Since several TNCSs have been discussed in the literature for spaces of dimension 3 [4, 7–10] and 4 [6, 9, 10], Zimba and Penrose's theorem shows how to create TNCSs for any dimension $k \geqslant 6$ (the $k = 5$ case is excluded since 5 cannot be obtained as the sum of some number of 3s and some number of 4s). In section 3 of this paper, we generalize to any finite dimension $n \geqslant 3$ ($n = 5$ included) the three-step procedure used by KS [4] to construct a TNCS in $n = 3$. The first two steps in this construction of TNCSs in section 3 are related to other variants of the BKS theorem, as we show in section 4; this reveals the relationship between different published proofs of the BKS theorem and allows us to classify most of them. Finally, in section 5, we discuss the physical meaning of the observables that appear in these proofs and the feasibility of joint measurements of compatible sets of them. The construction of other TNCSs is briefly discussed in the appendix.

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2. The Bell–Kochen–Specker theorem

In QM, physical observables and states can be represented by self-adjoint operators in a complex Hilbert space. Let $\hat{A}$ be a self-adjoint operator in a complex Hilbert space of dimension n, $\boldsymbol{H}^n$. $\hat{A}$ can always be expressed as a linear combination of a complete set $\{\hat{P}_i\}_{i=1}^n$ of projectors on one-dimensional orthogonal subspaces of $\boldsymbol{H}^n$, $\hat{A} = \sum_{i=1}^n a_i \hat{P}_i$. Physically, this means that the observable A can, in principle, be implemented as a set of yes–no (*propositions* [11]) 'one-dimensional' mutually compatible experiments, $\{P_i\}_{i=1}^n$.

In QM, the result $r(P_i)$ of a measurement of a proposition P_i is one of the eigenvalues of the projector $\hat{P}_i$ (1 with multiplicity 1 and 0 with multiplicity $n-1$). Since the identity operator in $\boldsymbol{H}^n$ can be decomposed into the sum of the projectors of a complete set, $\hat{I} = \sum_{i=1}^n \hat{P}_i$, the results of a joint measurement of all the 'one-dimensional' compatible propositions of a complete set $\{P_i\}_{i=1}^n$ must verify

$$\sum_{i=1}^{n} r(P_i) = 1\,. \tag{1}$$

Let us consider a NCHV theory in which propositions P_i can have predefined values $v(P_i)$ before the corresponding measurement; the BKS theorem asserts that, if $n \geqslant 3$, it is not possible to assign values $v(P_i)$ in an individual system to all the propositions P_i in such a way that this assignment verifies:

(i) Non-contextuality: each proposition is assigned a *single* value (1 or 0) which is the *same* independently of which other compatible propositions are chosen to form the complete set $\{P_i\}_{i=1}^n$.

(ii) The sum of the values assigned to any complete set of mutually compatible (one-dimensional) propositions $\{P_i\}_{i=1}^n$ verify

$$\sum_{i=1}^{n} v(P_i) = 1\,. \tag{2}$$

Let us formulate these constraints in another way. In the n-dimensional real projective space, $\boldsymbol{RP}^n$, we can identify one-dimensional projectors $\hat{P}_i$ with directions $\boldsymbol{r}_i$ (which we will represent as row vectors) on an n-dimensional unitary sphere with opposite points identified, by the relation $\hat{P}_i = \boldsymbol{r}_i^t \otimes \boldsymbol{r}_i$ (where t means transposition); commuting projectors (compatible propositions) correspond to orthogonal directions, and a complete set of n commuting projectors, to a set of n orthogonal directions.

From rules (i) and (ii) for the assignment of values to propositions, we can immediately deduce constraints for labelling directions with the values 1 or 0 (or equivalently with colours, for instance white and black, respectively). Following Zimba and Penrose [6], in this paper we assume that:

(A) No two orthogonal directions are both labelled 1 ('white').

(B) In any group of n mutually orthogonal directions, not all of the directions are labelled 0 ('black').

In short, given n orthogonal directions $\{\boldsymbol{r}_i\}$, one of them is labelled 1, $v(\boldsymbol{r}_j) = 1$, and the remaining ones 0, $v(\boldsymbol{r}_k) = 0$, $k \neq j$.

The BKS theorem can then be formulated as follows: *if $n \geqslant 3$, there are sets of directions that cannot be coloured in any way consistent with rules (A) and (B).* The construction of any such set proves the theorem.

3. Construction of some finite totally non-colourable sets for any $n \geqslant 3$

A *TNCS* is a set that cannot be labelled in *any way* using rules (A) and (B); the set proposed by KS [4] in dimension $n = 3$ is an example of this. In this section we will generalize their construction of a TNCS to any $\boldsymbol{RP}^n$ with $n \geqslant 3$.

The construction will proceed in three steps. In the first step we construct a set of directions such that a particular election of colour for one direction determines the colour of another one; we will use these sets in the second step to construct larger sets that are non-colourable for a particular election of colour for one of their directions; finally, this second type of sets will be used in the third step to construct still larger sets of directions that are non-colourable in any way.

3.1. Construction of a definite prediction set

Definition. We shall call a *definite prediction set* (DPS) a set $S = \{\boldsymbol{r}_k\}_{k=1}^{f}$ of directions such that for at least some election of value for a particular direction† $\boldsymbol{r}_1$ of S, the value for another direction $\boldsymbol{r}_f$ of S is determined, according to rules (A) and (B).

Lemma 1. In $\boldsymbol{RP}^n$, with $n \geqslant 3$, there are sets $S = \{\boldsymbol{r}_k\}_{k=1}^{f}$, such that $v(\boldsymbol{r}_1) = 1 \Rightarrow v(\boldsymbol{r}_f) = 1$ if $\boldsymbol{r}_1$ and $\boldsymbol{r}_f$ subtend an angle ϕ less than or equal to a certain value that we shall discuss in this section and also in the appendix; any such set is an example of DPS.

Proof. Let us explicitly construct one set with the above properties; to simplify the notation we will omit normalization coefficients in some of the vectors. Let $S = \{\boldsymbol{r}_k\}_{k=1}^{f=n+7}$, where $\boldsymbol{r}_1 = (1, 0, 0, 0, \ldots, 0)$, $\boldsymbol{r}_2 = (0, \cos\alpha, \sin\alpha, 0, \ldots, 0)$, $\boldsymbol{r}_3 = (\cot\phi, 1, -\cot\alpha, 0, \ldots, 0)$, $\boldsymbol{r}_4 = (\tan\phi \operatorname{cosec}\alpha, -\sin\alpha, \cos\alpha, 0, \ldots, 0)$; $\boldsymbol{r}_5$, $\boldsymbol{r}_6$ and $\boldsymbol{r}_7$ are obtained, respectively, from $\boldsymbol{r}_2$, $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$, replacing $\alpha \to \beta$, with $\alpha \neq \beta$; $\alpha, \beta \neq p\pi/2$, p integer; $\boldsymbol{r}_{n+5} = (\sin\phi, -\cos\phi, 0, 0, \ldots, 0)$, $\boldsymbol{r}_{n+6} = (0, 0, 1, 0, \ldots, 0)$, and $\boldsymbol{r}_{n+7} = (\cos\phi, \sin\phi, 0, 0, \ldots, 0)$. If $n \geqslant 4$, $\{\boldsymbol{r}_i\}_{i=8}^{n+4}$ are directions of the type $\boldsymbol{r}_i = (0, 0, 0, a_4, \ldots, a_n)$, where $a_{i-4} = 1$ and $a_j = 0$ if $j \neq i - 4$.

Figure 1 represents the directions of S and their orthogonality relations. Each point represents one direction, except for point $\{r_i\}_{i=8}^{n+4}$ which represents $n - 3$ directions (in the $n = 3$ case this point does not exist). Points on the same straight line represent mutually orthogonal directions. If, as we said before, we label the directions with value 1 'white' and the directions with value 0 'black', figure 1 represents *one* possible way of assigning values (colours) to every direction of S according to rules (A) and (B).

For $\boldsymbol{r}_4$ to be orthogonal to $\boldsymbol{r}_7$ (as represented in figure 1), it is necessary that

$$\sin\alpha \sin\beta \cos(\alpha - \beta) = -\tan^2\phi\,. \tag{3}$$

Since the left-hand side is bound between $-\frac{1}{8}$ and 1, then

$$|\phi| \leqslant \arctan(1/\sqrt{8})\,. \tag{4}$$

For each election of ϕ consistent with this inequality, α and β must verify (3).

With the orthogonality relations considered in figure 1, $v(\boldsymbol{r}_1) = 1$ (i.e. $\boldsymbol{r}_1$ labelled white) $\Rightarrow$ $v(\boldsymbol{r}_2) = v(\boldsymbol{r}_5) = \{v(\boldsymbol{r}_i)\}_{i=8}^{n+4} = v(\boldsymbol{r}_{n+6}) = 0$ by rule (A) (i.e. all these

† We will also admit as DPSs those which start with a particular election of values for *several* directions of S, provided that such an election can be implemented in some physical state. For example, if we prepare a system in a quantum state describable by the vector (1, 0, 0) (that means that $v(1, 0, 0) = 1$), then $v(0, 1, 0) = 0$ and also, for instance, $v(0, 1, 1) = 0$; therefore the election $v(0, 1, 0) = v(0, 1, 1) = 0$ is a valid starting election for a DPS. One of the reasons for including this possibility in the definition of DPS concerns the discussion regarding which is the 'minimum' DPS; see the footnote in section 4.1.

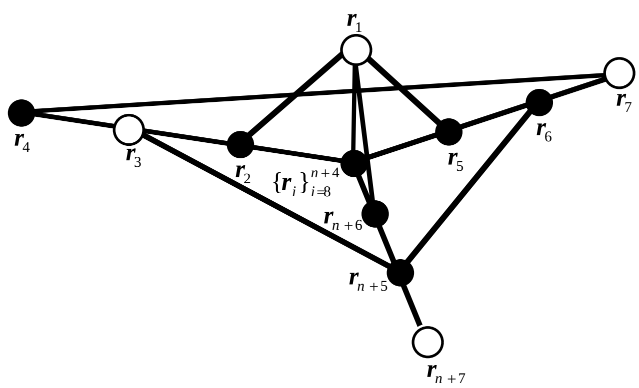

Figure 1. One possible way of assigning colours to a DPS with $n+7$ directions.

directions are black). Moreover, $v(\boldsymbol{r}_2) = \{v(\boldsymbol{r}_i)\}_{i=8}^{n+4} = 0 \Rightarrow v(\boldsymbol{r}_3) \neq v(\boldsymbol{r}_4)$, by rules (A) and (B). Suppose (as in figure 1) $v(\boldsymbol{r}_3) = 1$ then $v(\boldsymbol{r}_{n+5}) = 0$, by (A). Finally, $\{v(\boldsymbol{r}_i)\}_{i=8}^{n+6} = 0 \Rightarrow v(\boldsymbol{r}_{n+7}) = 1$, by (B). The other possibility (not represented in figure 1) is: $v(\boldsymbol{r}_3) = 0 \Rightarrow v(\boldsymbol{r}_4) = 1 \Rightarrow v(\boldsymbol{r}_7) = 0$, by (B) and (A), respectively. Therefore, $v(\boldsymbol{r}_5) = v(\boldsymbol{r}_7) = \{v(\boldsymbol{r}_i)\}_{i=8}^{n+4} = 0 \Rightarrow v(\boldsymbol{r}_6) = 1 \Rightarrow v(\boldsymbol{r}_{n+5}) = 0$, by (B) and (A). Again, $\{v(\boldsymbol{r}_i)\}_{i=8}^{n+6} = 0 \Rightarrow v(\boldsymbol{r}_{n+7}) = 1$. □

In short, using a set of $n+7$ directions we have proved that if $\boldsymbol{r}_1$ and $\boldsymbol{r}_f$ ($f \equiv n+7$) form an angle $|\phi| \leqslant \arctan(1/\sqrt{8})$, then $v(\boldsymbol{r}_1) = 1 \Rightarrow v(\boldsymbol{r}_f) = 1$. This set will be a building block for the following steps.

3.2. Construction of a partially non-colourable set

Definition. We shall call a *partially non-colourable set* (PNCS) a set T of directions such that there is *at least one election of value* for some direction of T that makes it impossible to assign values to the rest of directions of T according to rules (A) and (B).

Lemma 2. In $\boldsymbol{RP}^n$, with $n \geqslant 3$, PNCSs can be constructed by suitably 'chaining' several DPSs.

Proof. Let $S_1 = \{\boldsymbol{r}_{1k}\}_{k=1}^{f}$ be the set S of the previous section, with the particular election (as in [4]) $\phi = \pi/10$ (compatible with the constraint (4) for ϕ). Let S_j, $j = 2, \ldots, 5$, be obtained from S_{j-1} by a rotation of angle $\pi/10$ in the x–y plane (therefore, $\boldsymbol{r}_{21} = \boldsymbol{r}_{1f}, \ldots, \boldsymbol{r}_{51} = \boldsymbol{r}_{4f}, \boldsymbol{r}_{5f} = (0, 1, 0, \ldots, 0)$),

$$\boldsymbol{r}_{jk} = R\boldsymbol{r}_{(j-1)k} \tag{5a}$$

$$= R^{j-1}\boldsymbol{r}_{1k} \tag{5b}$$

where R is the following $n \times n$ matrix:

$$R = \begin{pmatrix} \cos(\pi/10) & -\sin(\pi/10) & 0 & \ldots & 0 \\ \sin(\pi/10) & \cos(\pi/10) & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}. \tag{6}$$

Finally, let us define $T = \{\boldsymbol{r}_{jk}\} = \{S_j\}_{j=1}^{5}$.

Applying the property of the set S five times, $v(\boldsymbol{r}_{11}) = 1 \Rightarrow v(\boldsymbol{r}_{1f})(\equiv v(\boldsymbol{r}_{21})) = 1 \Rightarrow v(\boldsymbol{r}_{2f})(\equiv v(\boldsymbol{r}_{31})) = 1 \Rightarrow \cdots \Rightarrow v(\boldsymbol{r}_{5f}) = 1$; but $\boldsymbol{r}_{11}$ and $\boldsymbol{r}_{5f}$ are orthogonal and therefore $v(\boldsymbol{r}_{11})$, $v(\boldsymbol{r}_{5f})$ cannot be simultaneously 1, according to (A). Then we conclude that T with the initial election $v(\boldsymbol{r}_{11}) = 1$ is non-colourable. □

Note that the other initial condition, $v(\boldsymbol{r}_{11}) = 0$, does not determine the final outcome $v(\boldsymbol{r}_{5f})$.

T contains $n+38$ different directions. This is so because from the number of directions obtained by multiplying 5 (the number of sets S in T) by $n+7$ (the number of directions in S), we have to subtract: four directions, because $\boldsymbol{r}_{j(n+7)} = \boldsymbol{r}_{(j+1)1}$, $j = 1, \ldots, 4$; another $4(n-2)$ directions, because $\boldsymbol{r}_{jk} = \boldsymbol{r}_{(j+1)k}$, with $j = 1, \ldots, 4$, and $k = 8, \ldots, n+4$ and $n+6$; and another direction because $\boldsymbol{r}_{11} = \boldsymbol{r}_{5(n+5)}$ (as can be checked using (5*b*)).

3.3. *Construction of a totally non-colourable set*

Definition. We shall call a *totally non-colourable set* (TNCS) a finite set of directions that cannot be coloured *in any way* according to rules (A) and (B).

Lemma 3. In $\boldsymbol{RP}^n$, $n \geqslant 3$, TNCSs can be constructed by suitably chaining several PNCSs.

Proof. Let $T_1 = \{\boldsymbol{r}_{1jk}\}$ be the set T of the proof of lemma 2, which linked the orthogonal directions $\boldsymbol{u}_1 = \{1, 0, 0, \ldots, 0\} = \boldsymbol{r}_{111}$ and $\boldsymbol{u}_2 = \{0, 1, 0, \ldots, 0\} = \boldsymbol{r}_{15(n+7)}$; let T_i be a similar set linking $\boldsymbol{u}_i$ with $\boldsymbol{u}_{i+1}$, and let us denote $U = \{\boldsymbol{r}_{ijk}\} = \{T_i\}_{i=1}^{n}$. Then

$$\boldsymbol{r}_{ijk} = P\boldsymbol{r}_{(i-1)jk} \tag{7a}$$

$$= P^{i-1}\boldsymbol{r}_{1jk} \tag{7b}$$

where P is the following $n \times n$ matrix:

$$P = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{8}$$

Using equations (5*b*) and (7*b*), we can write all the directions in the set $U = \{\boldsymbol{r}_{ijk}\}$ ($i = 1, \ldots, n$; $j = 1, \ldots, 5$; $k = 1, \ldots, n+7$) as

$$\boldsymbol{r}_{ijk} = P^{i-1}R^{j-1}\boldsymbol{r}_{11k} \tag{9}$$

where the directions $\{\boldsymbol{r}_{11k}\}_{k=1}^{n+7}$ are those of the original 'building block' $\{\boldsymbol{r}_k\}_{k=1}^{n+7}$ of section 3.1.

From the proof of lemma 2 and the identification $\boldsymbol{r}_{211} \equiv \boldsymbol{r}_{15(n+7)}$ we see that the election $v(\boldsymbol{r}_{111}) = 1$ implies $v(\boldsymbol{r}_{211}) = 1$, violating rule (A). However, if $v(\boldsymbol{r}_{111}) = 0$ then $v(\boldsymbol{r}_{211}) = 1$ can be 1 or 0. In the first case, $v(\boldsymbol{r}_{211}) = 1 \Rightarrow v(\boldsymbol{r}_{311}) = 1$ and that is, again, impossible by (A); so $v(\boldsymbol{r}_{211}) = 0$. Applying the same reasoning to all the directions in $\{\boldsymbol{r}_{i11}\}_{i=1}^{n}$, we conclude that $v(\boldsymbol{r}_{i11}) = 0$ for all $i = 1, \ldots, n$. But this is impossible by (B); therefore, U is a finite TNCS. □

U contains $39n$ different directions. This is so because from the number of directions obtained by multiplying n (the number of sets T in U) by $n+38$ (the number of directions in any T), we have to subtract $(n-1)n$ directions, because the set of directions $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)\} \in T_i$, $\forall i$, with $i = 1, \ldots, n$. In particular, for $n = 3$, U is the set of 117 directions considered by KS [4].

If $n \geqslant 4$, in addition to the orthogonality relations used in our proof, there are supplementary ones, but they do not interfere with the demonstration.

Starting from different DPSs (see the appendix), the same three-step procedure will allow us to construct other TNCSs.

4. Taxonomy of the proofs of the BKS theorem

4.1. The work of Gleason, Specker, Kochen and Bell

What we nowadays call the Bell–Kochen–Specker theorem, previously known simply as the Kochen–Specker theorem, was first stated by Specker [1] (although it can also be considered a corollary of a previous theorem proposed by Gleason [12]). The elementary geometrical arguments which proved the theorem were explicitly presented in later collaborations with Kochen [2–4]†.

Following a suggestion made by Jauch, Bell noted the transcendence of Gleason's theorem as an impossibility proof of NCHV (although Bell defended the physical plausibility of contextual hidden variables) [5]. KS's most cited paper [4], which was published just after (and without previous knowledge of) Bell's own work, contains many improvements: the theorem is formulated in terms of physical observables (instead of abstract projectors as in [5]) for whose joint measurement they suggested a specific procedure, and the proof involves a finite TNCS (our notation). A more detailed comparison between [4] and [5] can be found in [16].

As we have mentioned before, KS and Bell's proofs of the BKS theorem are different; moreover, ever since the publication of KS and Bell's papers, a great number of simplifications and variations of the original proofs have appeared. The following is an attempt to classify these proofs and explain how they relate to each other. For this purpose, the distinction we have introduced in section 3—between three types of sets (DPSs, PNCSs and TNCSs)—will be useful, since the different kinds of proofs are derived from the different types of sets.

4.2. Proofs derived from the sets with definite predictions

4.2.1. 'Continuum' proofs. With DPSs and the help of other geometrical arguments, we can obtain proofs of the BKS theorem without completing a finite non-colourable set (PNCS or TNCS). Bell [5], for instance, uses the following geometrical argument: since the directions can only be labelled 1 or 0, there must be two arbitrarily close directions with different values, and that is impossible. To justify this, let us use a DPS like the one we used to prove lemma 1 (Bell used one with 13 directions in $\boldsymbol{RP}^3$, see the appendix); in this set, if one direction has value 1, any other direction that forms with it an angle $\phi \leqslant \arctan(1/\sqrt{8})$ must also have value 1. Belinfante [17] uses a similar argument; he considers KS's DPS with eight directions in $\boldsymbol{RP}^3$ and an argument related to the relative size of the sets of directions with one or the other value. Also of this kind is the proof in [18]. These proofs have been called 'continuum proofs' [17, 19], where the word 'continuum' is used because it is

† In particular, an eight-direction DPS (our notation) for $n = 3$, now frequently used in the literature, appears for the first time in [2]. In its original form this set had 11 directions, but we can eliminate three since they do not play an essential role in the argument. Clifton [13] conjectures that this eight-direction DPS is the one with the least directions, and defies anyone to try to find a smaller one. In answer to Clifton, Vermaas [14] points out that one of the eight directions can also be eliminated; in fact, we could eliminate one more direction without losing physical significance (see the example in the previous footnote); Galindo [15] also points out that the set of eight directions 'can be improved to 6'.

assumed that, in order to implement those geometrical arguments, a continuum of directions are needed. The term ‘continuum proofs’ is criticized in [19]. KS objected to this type of argument and advocated ‘finite’ proofs, saying: ‘For otherwise a reasonable objection can be raised that in fact it is not physically meaningful to assume that there are a continuum number of quantum mechanical propositions’ [4].

4.2.2. ‘Probabilistic’ proofs. The DPSs also work for what we call here (as in [20]) ‘probabilistic’ proofs of the BKS theorem. In a DPS with a particular election for the value of the first direction, rules (A) and (B) assign certain values to some other directions; this assignation is inconsistent with certain statistical predictions of QM (see [21] for details). This possibility was first suggested by Stairs [22] (using KS’s eight-direction DPS). Recently, Clifton [13] has proposed similar arguments (with the same eight-direction DPS, and the 13-direction DPS used by Bell); some minor mistakes in [13] have been corrected by other authors [14, 23]). In [21] we examined another DPS with 14 directions (see also the appendix) which shows even greater discrepancies with QM, and we discussed how to obtain a simple experimental test between QM and NCHV theories.

Certain proofs of the Bell theorem ‘without inequalities’ but ‘with probabilities’ [24–26] can also be interpreted as probabilistic proofs of the BKS theorem [25].

4.3. Partially non-colourable sets and ‘state-specific’ proofs

With PNCSs we obtain contradictions between rules (A) and (B), starting from a particular election of value for the first direction of the set. Since $\boldsymbol{RP}^n$ has spherical symmetry, and the first direction has been chosen arbitrarily (and there must be at least one direction with that value), the contradiction which we arrived at using a PNCS is enough to prove BKS’s theorem (in particular, there is no need to construct a TNCS). This allows us to greatly simplify the proof by reducing the number of directions implicated in it. This is the type of proof presented by Friedeberg [27] (with an unspecified number of directions in $\boldsymbol{RP}^3$), Peres and Ron [28] (whose set of 109 directions in $\boldsymbol{RP}^3$ contains *two* PNCSs, one of 71, and another of 40 directions), one of the authors [29] (with 38 directions in $\boldsymbol{RP}^3$), and Kernaghan and Peres [20] (with 13 directions in $\boldsymbol{RP}^8$).

Certain proofs of the Bell theorem ‘without probabilities’ [30–33] and also [34, 35] (with a recursive definition for elements of reality) admit a reading as ‘state-specific’ proofs of the BKS theorem.

4.4. Totally non-colourable sets and ‘state-independent’ proofs

Some TNCSs, in order of publication, are those presented by KS with 117 directions in $\boldsymbol{RP}^3$ [4], de Obaldia, Shimony and Wittel’s with 138 directions in $\boldsymbol{RP}^3$ [7], Conway and Kochen’s [8] with 31 directions in $\boldsymbol{RP}^3$, those of Peres with 33 directions in $\boldsymbol{RP}^3$ and 24 directions in $\boldsymbol{RP}^4$ [9], those of Penrose with 33 directions in $\boldsymbol{CP}^3$ (three-dimensional complex projective space) and 40 directions in $\boldsymbol{CP}^4$ [10], Zimba and Penrose’s with 28 directions in $\boldsymbol{CP}^4$ [6] (a subset of Penrose’s [10]), Kernaghan’s with 20 directions in $\boldsymbol{RP}^4$ [36] (a subset of Peres’ [9]), and Kernaghan and Peres’ with 36 directions in $\boldsymbol{RP}^8$ [20].

TNCSs lead to what, following [20], we will call ‘state-independent’ proofs of the BKS theorem. ‘State-specific’ proofs like [31, 32, 34, 35] can be completed to generate state-independent proofs, involving ‘multi-dimensional’ spin operators of two or three spin-$\frac{1}{2}$ particles [37]; see also [19, 34]. The generalization to n spin-$\frac{1}{2}$ particles of these state-independent arguments [38, 39], would allow us to obtain TNCSs of one-dimensional propositions in Hilbert spaces of dimension 2^n. This was presumably the way followed in [9] and [20].

BKS's theorem against NCHV can also be extended [22, 40] to a more restrictive class of hidden variables, the 'local-non-contextual' hidden variables (using a terminology introduced by Shimony [41]). In the heart of such statements lies the same geometrical argument treated in this paper.

5. Physical interpretation of the proofs

Up until now we have focused our attention on the mathematical aspects of the problem, deliberately avoiding two physically relevant questions:

(i) What is the interpretation in terms of physical propositions of the projection operators used in the proofs?
(ii) How can we simultaneously measure any complete set of such propositions?

The object of this section is to answer both of these questions. First, we review the physical interpretations used in the literature for some of the proofs, and then we shall see that, although in general the answer to the above questions is not trivial, the measurements involved in some physical systems described by Hilbert spaces of arbitrary finite dimension $n \geqslant 3$ are in principle feasible.

KS [4] pointed out that in $n = 3$ there is a bijective map between the set of directions $\boldsymbol{r}_i$ in $\boldsymbol{RP}^3$ and the square of the spin component in that direction of a spin-1 particle, S_i^2. The relation is $\boldsymbol{r}_i^t \otimes \boldsymbol{r}_i = \hat{I} - \hbar^{-2} S_i^2$, and has the following useful property: $[S_i^2, S_j^2] = 0 \Leftrightarrow \boldsymbol{r}_i \cdot \boldsymbol{r}_j = 0$. We can measure an individual S_i^2 using ordinary Stern–Gerlach devices, and a joint measurement of a complete set of these observables (a joint measurement of S_x^2, S_y^2 and S_z^2, for example) can be achieved using an electromagnetic field with orthorhombic symmetry; see [4, 42]. The KS theorem then proves that it is not possible to assign NCHV values consistently to a finite set of observables S_i^2.

In $n = 4$, Penrose [6, 10] found a non-colourable set of 40 directions, at least 20 of which can be identified with projections of the type $\hat{P}_i = |S_i = \hbar/2\rangle\langle S_i = \hbar/2|$, where S_i represents the spin component of a spin-$\frac{3}{2}$ particle in the direction $\boldsymbol{r}_i$ (these projections have the property: $[\hat{P}_i, \hat{P}_j] = 0 \Leftrightarrow \boldsymbol{r}_i \cdot \boldsymbol{r}_j = \frac{1}{3}$).

Other non-colourable sets in $n = 4$ [9] and $n = 8$ [20] have been obtained from sets of 'multi-dimensional' projectors (in contradistinction to the one-dimensional 'propositions' considered until now); these sets are also non-colourable according to rules similar to (A) and (B) [9, 34, 43]. These observables are obtained as tensor products of spin components of several spin-$\frac{1}{2}$ particles (two particles for $n = 4$ [9, 34], three for $n = 8$ [19, 37]). The physical interpretation of these observables is straightforward, but the question of their joint measurement has not yet been solved [35].

Our answer to both questions in the case of Hilbert spaces of arbitrary finite dimension is based on two results. The first, a well known theorem of von Neumann [44] and Varadarajan [45], asserts that given a set $\{O_i\}$ of compatible observables, there exists an observable O and a set of Borel functions $\{f_i\}$ such that $O_i = f_i(O)$. In particular, if $\{O_i\}$ is a complete set of compatible observables, the observable O is maximal (non-degenerate). But as Jauch points out [46], 'The significance of this theorem is more mathematical than practical. The reason is that it is often easy to describe physical arrangements which measure a set of commuting observables, while it may be practically impossible to describe such an arrangement for the observable O of which they are all functions'.

Swift and Wright [47] provide the second important result for our purpose: 'Modulo the ability to create in the laboratory any electromagnetic field consistent with Maxwell's equations, ... using a generalized Stern–Gerlach apparatus, every Hermitian operator acting on the Hilbert space of a spin-s particle can be measured and a beam of particles can be produced in the state corresponding to any given ray in the Hilbert space'.

Both of these results provide a solution to the initial questions raised in this section. Identifying $\boldsymbol{H}^n$ with the Hilbert space of the spin states of a *single particle* of spin $s = (n-1)/2$ provides us with a physical example in which any complete set of propositions that appear in the proofs of the BKS theorem for any dimension $n \geqslant 3$ can be expressed (von Neumann and Varadarajan) in terms of a single maximal observable, which in turn can be measured, at least in principle (Swift and Wright), using a suitable generalized Stern–Gerlach apparatus.

6. Concluding remarks

This paper has shown a way to obtain physically meaningful (see the end of section 5), finite, state-independent [20] proofs of the BKS theorem in any dimension n. Our three-step construction, which is a generalization of [4], has the advantage of being essentially the same for any $n \geqslant 3$, and fills the gap left by Zimba and Penrose's result [6]. But on the other hand, the TNCSs obtained using this procedure are not the most 'economical': for particular dimensions, TNCSs with less than $39n$ directions are known (see section 4.4 and the appendix). This point raises an interesting, although strictly mathematical problem, as yet still unsolved: what is the minimum number of directions in each dimension, $N(n)$, necessary to have a TNCS in that dimension?

More interesting from the physical point of view is the way, initiated by Heywood and Redhead [40], of applying BKS geometrical arguments plus a locality criterion to composite systems. More recent works by Greenberger–Horne–Zeilinger [30, 33], Mermin [19, 31, 32, 37], Peres [34, 35], Hardy [24] and others, provide a deeper insight in this direction.

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Appendix. Other DPSs, PNCSs and TNCSs

Figures A1 and A2 represent two other possible DPSs in $\boldsymbol{RP}^n$ such that $v(\boldsymbol{r}_1) = 1 \Rightarrow v(\boldsymbol{r}_f) = 1$. The conventions are the same as used in figure 1. The initial election is $v(\boldsymbol{r}_1) = 1$ ('white'). Figure A1 (figure A2) reflects one of the four (five) possible ways to assign values (colours) to the remaining directions of the set.

The DPS in figure A1 has $f = n + 10$ directions. Directions $\boldsymbol{r}_1$ to $\boldsymbol{r}_7$ are the same as in section 3.1; the expressions of $\boldsymbol{r}_{n+8}$ to $\boldsymbol{r}_{n+10}$ coincide, respectively, with those of $\boldsymbol{r}_{n+5}$ to $\boldsymbol{r}_{n+7}$ in section 3.1. If $n \geqslant 4$, directions $\{\boldsymbol{r}_i\}_{i=11}^{n+7}$ coincide, respectively, with $\{\boldsymbol{r}_i\}_{i=8}^{n+4}$ in section 3.1. Three new directions appear, $\boldsymbol{r}_8 = (0, 1, 0, 0, \ldots, 0)$, $\boldsymbol{r}_9 = (\sin\alpha\cos\alpha, 0, -\tan\phi, 0, \ldots, 0)$, and $\boldsymbol{r}_{10}$, which is obtained from $\boldsymbol{r}_9$ with the change $\alpha \rightarrow \beta$, where $\alpha \neq \beta \neq p\pi/2$, with p integer. Similarly, as in section 3.1, $\boldsymbol{r}_9 \perp \boldsymbol{r}_{10} \Rightarrow |\phi| \leqslant \arctan\left(\frac{1}{2}\right)$.

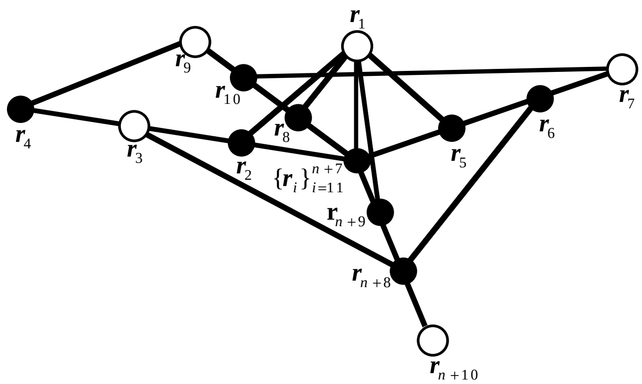

Figure A1. One possible way of assigning colours to a DPS with $n+10$ directions.

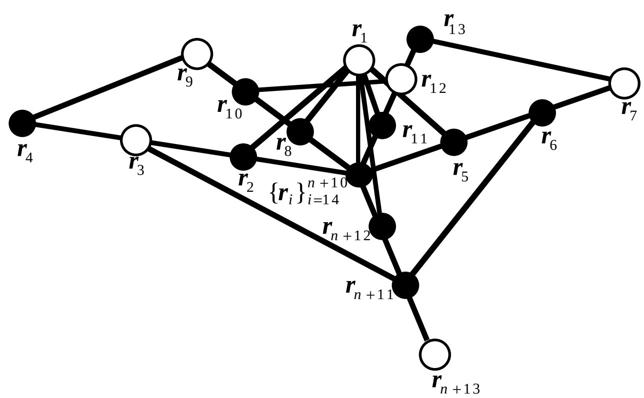

Figure A2. One possible way of assigning colours to a DPS with $n+13$ directions.

The DPS in figure A2 has $f = n + 13$ directions. The form of directions $\boldsymbol{r}_1$ to $\boldsymbol{r}_9$ coincides with those in the previous paragraph; $\boldsymbol{r}_{n+11}$ to $\boldsymbol{r}_{n+13}$ with $\boldsymbol{r}_{n+5}$ to $\boldsymbol{r}_{n+7}$ in section 3.1, respectively. If $n \geqslant 4$, directions $\{\boldsymbol{r}_i\}_{i=14}^{n+10}$ coincide with $\{\boldsymbol{r}_i\}_{i=8}^{n+4}$ in section 3.1. The remaining directions are $\boldsymbol{r}_{10} = (\tan\phi, 0, \sin\alpha\cos\alpha, 0, \ldots, 0)$, $\boldsymbol{r}_{11} = (0, \cos\gamma, \sin\gamma, 0, \ldots, 0)$, with $\alpha \neq \beta \neq \gamma \neq p\pi/2$, with p integer; $\boldsymbol{r}_{12} = (\tan\phi\,\mathrm{cosec}\,\beta\ \sec(\beta-\gamma), -\sin\gamma, \cos\gamma, 0, \ldots, 0)$, and $\boldsymbol{r}_{13} = (\cot\phi\sin\beta\cos(\beta-\gamma), \sin\gamma, -\cos\gamma, 0, \ldots, 0)$. $\boldsymbol{r}_{10} \perp \boldsymbol{r}_{12} \Rightarrow |\phi| \leqslant \arctan(3^{3/4}/4)$.

In the $n = 3$ case, the DPS in figure 1 is the one with 10 directions proposed by KS in [4]; the set figure A1 is the one with 13 directions proposed by Bell in [5]; and the set in figure A2 is a DPS with 16 directions which has as a subset the DPS with 14 directions considered in [21]†.

An interesting property of these sets is that, at the expense of just a few more directions, the angle between the initial and final directions, $\boldsymbol{r}_1$ and $\boldsymbol{r}_f$, for which $v(\boldsymbol{r}_1) = 1 \Rightarrow v(\boldsymbol{r}_f) = 1$, is greater than in the DPS of figure 1, namely $|\phi| \leqslant \arctan\left(\frac{1}{2}\right) = 0.464$ in the set of figure A1 and $|\phi| \leqslant \arctan(3^{3/4}/4) = 0.518$ in the

† If we eliminate the last two directions in the DPSs of figures 1, A1, A2, we have three other DPSs (with $f' = n+5$, $n+8$ and $n+11$, respectively) such that $v(\boldsymbol{r}_1) = 1 \Rightarrow v(\boldsymbol{r}_{f'}) = 0$.

set of figure A2, versus $|\phi| \leqslant \arctan(1/\sqrt{8}) = 0.340$ in the set of figure 1. This is useful in some of the 'probabilistic' versions of the BKS theorem, as it allows a greater discrepancy with QM [13, 21].

These DPSs could also be used to construct other PNCSs (and then other TNCSs). Some of them have a smaller number of directions than the one described in section 3.2 (and then in 3.3). For instance, one could chain successively two of the sets in figure A1, choosing in both $\phi = \arctan\left(\frac{1}{2}\right)$ (and therefore $\alpha = -\beta = \pi/4$), and then two of the sets of section 3.1, the first with $\phi = (2\pi/5) - 2\arctan(\frac{1}{2})$, and the second with $\phi = \pi/10$, to obtain a PNCS with $n + 34$ directions (instead of $n + 38$) and then a TNCS. An explicit calculation shows that this TNCS has 96 directions if $n = 3$, 136 directions if $n = 4$, and $35n$ directions if $n \geqslant 5$ (instead of $39n$ for any n).

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